Analysis 1, Summer 2023

List 1

Sequences, limits of sequences

20. If $a_n = (n+2)^3$, give the value of a_3. $5^3 = \boxed{125}$

21. For the sequence $b_n = n^{-n}$, what are the values b_1, b_2, and b_3?

    $\boxed{b_1 = 1}$, $\boxed{b_2 = \frac{1}{4} = 0.25}$, $\boxed{b_3 = \frac{1}{27} \approx 0.037037}$

22. If $c_n = (1+\frac{1}{n})^n$, what are the values c_1, c_2, and c_3? Give exact formulas (by hand) and decimal answers (using a calculator). $\boxed{c_1 = 2}$, $\boxed{c_2 = \frac{9}{4} = 2.25}$, $\boxed{c_3 = \frac{64}{27} \approx 2.3704}$

23. For the sequence $a_n = n^2 - 1$, give a formula for a_{n+1}.
    $(n+1)^2 - 1 = (n^2 + 2n + 1) - 1 = \boxed{n^2 + 2n}$

24. Consider the sequence

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 22 \\ a_3 &= 222 \\ a_4 &= 2222 \\ a_n &= \underbrace{22...2}_{n \text{ digits}} \end{aligned}$$

    (a) Calculate $(10a_1 + 2) - a_1$, then $(10a_2 + 2) - a_2$, then $(10a_3 + 2) - a_3$.
    $\boxed{20, \ 200, \ 2000}$

    (b) Find a formula for $(10a_n + 2) - a_n$ in terms of n only. $\boxed{2 \cdot 10^n}$

    (c) Find a formula for a_n. $\boxed{\frac{2}{9}(10^n - 1)}$

> The sequence a_n **converges** to the real number L if for any $\varepsilon > 0$ there exists an N such that
> $$L - \varepsilon < a_n < L + \varepsilon \qquad \text{for all } n > N.$$
> In this case we say the **limit** of the sequence is L, and we write
> $$\lim_{n\to\infty} a_n = L.$$
> A sequence that does not converge to any number is said to **diverge**.

25. (a) For which positive integers n is $4 - \frac{1}{100} < \frac{8n}{2n+9} < 4 + \frac{1}{100}$?
    $\boxed{n \geq 1796}$

    (b) For which positive integers n is $\frac{8n}{2n+9} = 4$? $\boxed{\text{None!}}$

    (c) Is it true that $\lim_{n\to\infty} \frac{8n}{2n+9} = 4$? $\boxed{\text{Yes}}$

26. Calculate $\lim_{n\to\infty} \frac{3n^2 + n + \sqrt{n}}{5n^2} = \boxed{\frac{3}{5}}$

27. Determine whether each sequence converges or diverges.

   (a) n^n $\boxed{\text{diverges}}$

   (b) $\frac{n}{n+1}$ $\boxed{\text{converges}}$

   (c) $(-1)^n$ $\boxed{\text{diverges}}$

   ☆(d) $\sin(3n)$ $\boxed{\text{diverges}}$

   (e) $\sin(\pi n)$ $\boxed{\text{converges}}$ because the sequence is $0, 0, 0, 0, \ldots$

   (f) $\frac{(-1)^{n+1}}{n^n}$ $\boxed{\text{converges}}$ Specifically, this converges to 0.

> We say a_n **diverges to infinity** and write $\lim_{n\to\infty} a_n = \infty$ if for any $M > 0$ there exist an N such that
>
> $$a_n > M \qquad \text{for all } n > N.$$
>
> Similarly, we write $\lim_{n\to\infty} a_n = -\infty$ if for any $M > 0$ there exist an N such that
>
> $$a_n < -M \qquad \text{for all } n > N.$$

28. Find the following limits if they exist.

   (a) $\lim_{n\to\infty} \frac{n+13}{n^2} = \boxed{0}$

   (b) $\lim_{n\to\infty} \frac{(n+5)(n-2)}{n^2 - 6n + 7} = \boxed{1}$

   (c) $\lim_{n\to\infty} \frac{n^2}{n+13} = \boxed{\infty}$

   (d) $\lim_{n\to\infty} -2^n = \boxed{-\infty}$

   (e) $\lim_{n\to\infty} (-2)^n$ $\boxed{\text{doesn't exist}}$

   (f) $\lim_{n\to\infty} 2^{-n} = \boxed{0}$

   ☆(g) $\lim_{n\to\infty} 2^{1/n} = \boxed{1}$

29. Find $\lim_{n\to\infty} \left((9\sqrt{n} + \frac{1}{\sqrt{n}})^2 - 81n\right) = \boxed{18}$

☆30. Find $\lim_{n\to\infty} n \cdot (2^{1/n} - 1)$. The ☆ means that this task is harder than what is normally expected in this course. $\boxed{\ln(2)}$

31. (a) Simplify the formula $\frac{\left(\sqrt{n} - \sqrt{n-1}\right)\left(\sqrt{n} + \sqrt{n-1}\right)}{\sqrt{n} + \sqrt{n-1}} = \boxed{\frac{1}{\sqrt{n} + \sqrt{n-1}}}$

   (b) Find $\lim_{n\to\infty} \sqrt{n} - \sqrt{n-1} = \lim_{n\to\infty} \frac{1}{\sqrt{n} + \sqrt{n-1}} = \boxed{0}$

32. Use the Squeeze Theorem with $\dfrac{-1}{n} \le \dfrac{\cos(n)}{n} \le \dfrac{1}{n}$ to find $\lim\limits_{n\to\infty} \dfrac{\cos(n)}{n}$.

$\lim\limits_{n\to\infty} \dfrac{-1}{n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, so by Squeeze Theorem we have $\lim\limits_{n\to\infty} \dfrac{\cos(n)}{n} = \boxed{0}$.

☆33. Use the fact that $\left(1 - \dfrac{1}{\sqrt{n}}\right)^n \le \dfrac{1}{n}$ to find $\lim\limits_{n\to\infty} (1/n)^{1/n}$.

We need an inequality involving $(1/n)^{1/n}$, but the right side of $\left(1 - \frac{1}{\sqrt{n}}\right)^n \le \frac{1}{n}$ is just $(1/n)$. Raising both sides of the equation to the power $1/n$ gives

$$1 - \frac{1}{\sqrt{n}} \le \left(\frac{1}{n}\right)^{1/n}.$$

The Squeeze Theorem requires two inequalities. The left-hand side now has limit

$$\lim_{n\to\infty} 1 - \frac{1}{\sqrt{n}} = 1 - 0 = 1,$$

so another inequality involving a limit of 1 would be good. In fact,

$$\left(\frac{1}{n}\right)^{1/n} \le 1$$

is enough, and it is true because $\frac{1}{n} \le 1^n$ is true for all $n \ge 1$ (this is just $\frac{1}{n} \le 1$). We can now use the Squeeze Theorem:

$$1 - \frac{1}{\sqrt{n}} \le \left(\frac{1}{n}\right)^{1/n} \le 1$$

$$\lim_{n\to\infty} 1 - \frac{1}{\sqrt{n}} \le \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} \le \lim_{n\to\infty} 1$$

$$1 \le \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} \le 1$$

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \boxed{1}$$

34. (a) The *definition* of the number "0.385" is

$$3 \cdot 10^{-1} \ + \ 8 \cdot 10^{-2} \ + \ 5 \cdot 10^{-2}.$$

Write this number as a fraction (or an integer, if possible). $\boxed{\dfrac{385}{1000} \text{ or } \dfrac{77}{200}}$

(b) The *definition* of the number "0.2222..." is the ***limit*** of the sequence

$$S_1 = 0.2$$
$$S_2 = 0.22$$
$$S_3 = 0.222$$
$$S_4 = 0.2222$$
$$S_n = 0.\underbrace{22...2}_{n \text{ digits}}$$

Write this number as a fraction (or an integer, if possible).
Hint: See Task 24(c).

$S_n = \frac{a_n \text{ from Task 24(c)}}{10^n} = \frac{\frac{2}{9}(10^n - 1)}{10^n} = \frac{2}{9}(1 - 10^{-n})$.

Therefore $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{9}(1 - 10^{-n}) = \boxed{\frac{2}{9}}$

(c) The *definition* of the number "0.9999..." is the ***limit*** of the sequence

$$S_n = 0.\underbrace{99...9}_{n \text{ digits}}.$$

Write this number as a fraction (or an integer, if possible).

$S_n = 1 - 10^{-n}$, so $\lim_{n\to\infty} S_n = \boxed{1}$